Student number: ________________

2023

YEAR 12
Trial
EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 10 minutes
- Working time - 120 minutes
- Write using black or blue pen
- NESA approved calculators may be used
- A reference sheet is provided at the back of this paper

Total marks: 70

Section I – 10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section
- Answer multiple choice questions by completely colouring in the appropriate circle on the answer sheet provided

Section II – 60 marks

- Attempt questions 11 – 16 in the answer booklet provided
- Allow about 1 hour and 45 minutes for this section
- Start each question on a new page showing all relevant mathematical reasoning and/or calculations

Section I

10 marks

Attempt Questions 1-10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10 .

Question 1 (1 mark)

Which of the following is the remainder when the polynomial $P(x) = (x+2)^3 + 2$ is divided by $(x-2)$?

A. 2

B. 66

C. $x-2$

D. $x+2$

Question 2 (1 mark)

What is the value of $\lim_{x \to 0} \left(\frac{sin\frac{1}{3}x}{2x} \right)$

A. $\frac{1}{6}$

B. $\frac{2}{3}$

C. $\frac{3}{2}$

D. 6

Question 3 (1 mark)

After t years the number N of individuals in a population is given by $N = 400 + 100e^{-0.1t}$. What is the difference between the initial population size and the limiting population size?

A. 100

B. 300

C. 400

D. 500

Question 4 (1 mark)

What is the acute angle between the vectors $\underset{\sim}{i} + 2\underset{\sim}{j}$ and $4\underset{\sim}{i} + 2\underset{\sim}{j}$ correct to the nearest degree?

A. 18^o

B. 26^o

C. 32^o

D. 37^o

Question 5 (1 mark)

A school commitee consists of 8 members and a chairperson. The members are selected from 12 students. The chairperson is selected from 4 teachers. In how many ways could the committee be selected?

A. ${}^{12}C_8 +^4 C_1$

B. ${}^{12}P_8 +^4 P_1$

C. ${}^{12}C_8 \times^4 C_1$

D. ${}^{12}P_8 \times^4 C_1$

Question 6 (1 mark)

The vectors $\underset{\sim}{p} = 4\underset{\sim}{i} + (a+1)\underset{\sim}{j}$ and $\underset{\sim}{q} = a\underset{\sim}{i} - 2\underset{\sim}{j}$ are perpendicular. What is the value of a?

A. -1

B. 1

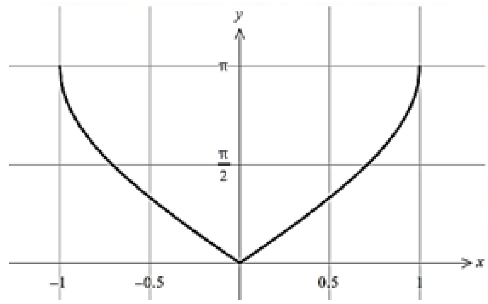
C. $\frac{1}{3}$

D. $-\frac{1}{3}$

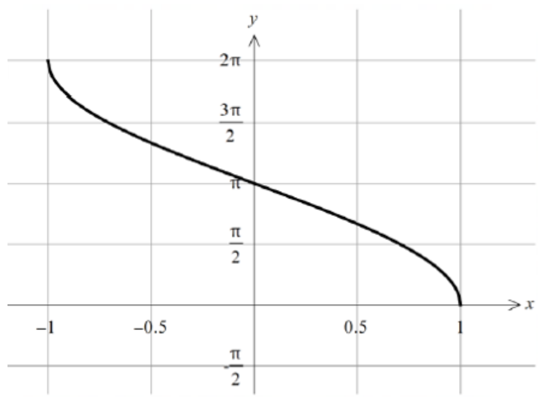
Question 7 (1 mark)

Which graph best represents $y = |2cos^{-1}x - \pi|$?

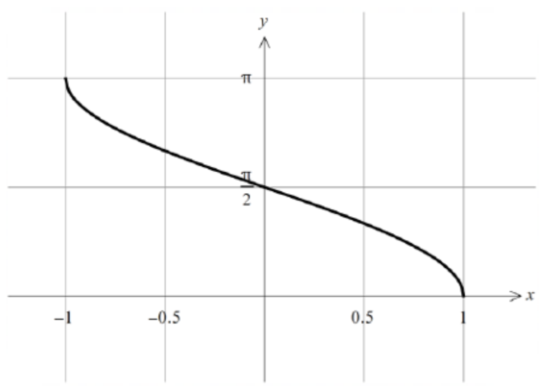
A.



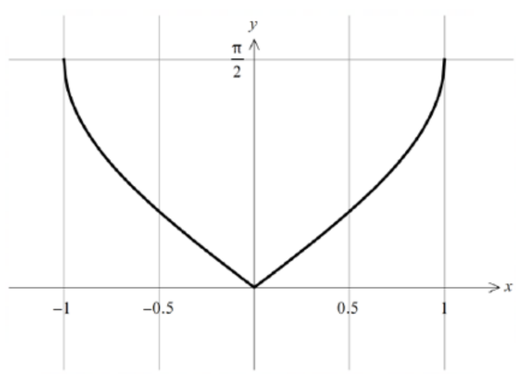
B.



C.



D.



Question 8 (1 mark)

A vertical tower AO of height h metres stands with its base A on horizontal ground. A stone is projected horizontally from the top O of the tower with speed Vms^{-1}. The stone moves in a vertical plane under gravity where the acceleration due to gravity is gms^{-2}. At time t seconds its position vector relative to O is $\underset{\sim}{r}(t) = (Vt)\underset{\sim}{i} - \left(\frac{1}{2}gt^2\right)\underset{\sim}{j}$. The stone hits the ground at an angle of 45^o to the horizontal. What is the time of the impact?

A. $\dfrac{V}{2g}$ seconds

B. $\dfrac{V}{g}$ seconds

C. $\dfrac{2V}{g}$ seconds

D. $\dfrac{4V}{g}$ seconds

Question 9 (1 mark)

$g(x)$ is the inverse function of $f(x) = e^{x-1}$. Which one of these statements must be true for all x in the domain of $g(x)$?

A. $g(x) > 0$

B. $g(x) < 0$

C. $g''(x) < 0$

D. $g'(x) < 0$

Question 10 (1 mark)

$\sin(3x + x) - \sin(3x - x) =$

A. $-2\sin 3x \sin x$

B. $2\cos 3x \sin x$

C. $2\cos 3x \cos x$

D. $2\sin 3x \sin x$

Section II

60 marks

Attempt Questions 11-16

Allow about 1 hour and 45 minutes for this section

Answer each question on a separate page in the writing booklet.

Extra writing booklets are available

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (10 marks)

(a) Solve the inequality $\dfrac{2x-1}{x+2} > 1$ [3]

(b) Use the substitution $u = 6 - x$ to find the exact value of $\displaystyle\int_1^6 x\sqrt{6-x}\,dx$ [3]

(c) A group of 40 new Year 7 students is going to be randomly divided into two classes A and B of 20 students each. Four of the students, Anne, Ben, Carrie and David have been together in the same class since they started school and are close friends.

i. What is the probability that the four friends will be in the same class? [2]

ii. What is the probability that exactly three of the friends will be in the same class? [2]

Question 12 (10 marks)

(a) Find the exact value of $\displaystyle\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$ [3]

(b) Find the cartesian equation for the curve with parametric equations $x = 1 + 2\cos 2t$ and $y = 2 + 2\sin 2t$ [2]

(c) Prove by principle of mathematical induction that $7^n - 3^n$ is divisible by 4 for $n \geq 1$. [3]

(d) Prove the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ [2]

Question 13 (11 marks)

(a) If $t = \tan\dfrac{x}{2}$, show that $\dfrac{1+\cos x+\sin x}{1-\cos x+\sin x} = \cot\dfrac{x}{2}$. [3]

(b) Using $t-$ formulae solve the equation $\sin\theta + \cos\theta = \dfrac{1}{2}$ for $[0, 2\pi]$. Answer in radians to 3 decimal places. [4]

(c) i. Express $\cos x - \sqrt{3}\sin x$ in the form $R\cos(x+\alpha)$, where α is an acute angle. [2]

ii. Hence, solve $\cos x - \sqrt{3}\sin x = 1$ for $[-\pi, \pi]$ [2]

Question 14 (10 marks)

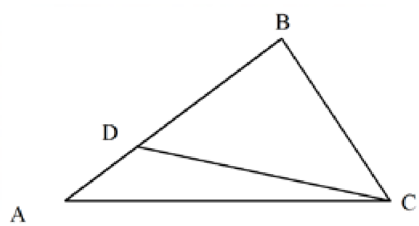
(a) If α, β and γ are roots of $x^3 - 5x^2 + 7x + 5 = 0$

i. Find $\alpha + \beta + \gamma$ [1]

ii. Find $\alpha\beta + \beta\gamma + \gamma\alpha$ [1]

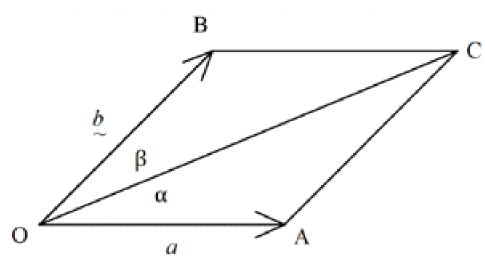
(b) Given $\underset{\sim}{u} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{v} = -2\underset{\sim}{i} + 4\underset{\sim}{j}$. Find $proj_{\underset{\sim}{u}}\, \underset{\sim}{v}$ [2]

(c) In ΔABC , D is a point on AB, where $|\overrightarrow{AD}| : |\overrightarrow{DB}| = 2{:}3$ [2]



Given $\overrightarrow{AD} = \underset{\sim}{a}$, $\overrightarrow{AC} = \underset{\sim}{b}$ and $\overrightarrow{CB} = \underset{\sim}{c}$. Show that $\underset{\sim}{b} = \dfrac{1}{2}(5\underset{\sim}{a} - 2\underset{\sim}{c})$

(d) $OACB$ is rhombus. $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$, $\angle AOC = \alpha$ and $\angle COB = \beta$.



i. For the vectors in the diagram above prove that $\underset{\sim}{a}.(\underset{\sim}{a} + \underset{\sim}{b}) = \underset{\sim}{b}.(\underset{\sim}{a} + \underset{\sim}{b})$ [2]

ii. Hence, prove diagonal OC bisects $\angle AOB$. [2]

Question 15 (10 marks)

(a) Jerry stands at the top of a building $30m$ tall and throws a ball with a velocity of $15ms^{-1}$ at an angle of 45^o above the horizontal. The ball eventually reaches the ground.

i. Derive the vector displacement of the ball in the form $\underset{\sim}{s} = x\underset{\sim}{i} + y\underset{\sim}{j}$. You may assume that $g = 10ms^{-2}$. [3]

ii. Find the time taken for the ball to reach the ground. [1]

(b) Consider the funstion $f(x) = \sin^{-1}(x-1)$.

i. Find the domain of the function. [1]

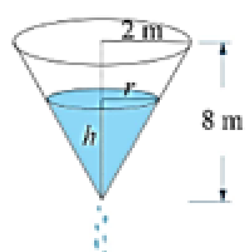
ii. Sketch the graph of the curve $y = f(x)$ showing the endpoints and the x-intercept(s). [2]

iii. The region in the first quadrant bounded by the curve $y = f(x)$ and the y-axis between the lines $y = 0$ and $y = \dfrac{\pi}{2}$ is rotated through one complete revolution about the y-axis. Find in simplest exact form the volume of the solid of revolution. [3]

Question 16 (9 marks)

(a) Find the constant term in the expression $\left(x + \dfrac{2}{x}\right)^6$ [3]

(b) An inverted conical container is 8 metres deep and has a base radius of 2 metres. Water is leaking from the container at a constant rate of $\dfrac{dV}{dt} = 0.1$ metres/hour, where V is the volume of the water in the container. Assume the container is full initially.



i. Show that $V = \dfrac{\pi}{48}h^3$, where h is the height of the remaining water in the container. [1]

ii. Hence, find the height of water in the container when $\dfrac{dh}{dt} = 0.02$ metres/hour, correct to two decimal places. [2]

Question 16 continued on next page...

Question 16 (continued)

(c) Newton's Law of cooling states that when an object at temperature T^oC is placed in an environment at temperature T_0^o, the rate of temperatre loss is given by the equation

$$\frac{dT}{dt} = k(T - T_o)$$

Where t is the time in seconds and k is a constant.

A packet of peas, initially at 24^oC, is placed in a snap freeze refrigerator in which the internal temperature is maintained at -40^oC.

i. Show that $T = -40 + 64e^{kt}$ satisfies the cooling equation $\dfrac{dT}{dt} = k(T - T_o)$. [1]

ii. After 5 seconds, the temperature of the packet is 19^oC. How long will it take for the temperature of the packet to reduce to 0^oC? [2]

End of paper

1) $P(x) = (x+2)^3 + 2$

Rem $= P(2) = (2+2)^3 + 2$

$= 66$

Ans. B.

2) $\lim\limits_{x \to 0} \left(\dfrac{\sin(\frac{1}{3}x)}{2x} \right)$

$= \frac{1}{6} \lim\limits_{x \to 0} \left(\dfrac{\sin \frac{1}{3}x}{\frac{1}{3}x} \right)$

$= \frac{1}{6} \times 1$

$= \frac{1}{6}$

Ans. A

3) $N = 400 + 100e^{-0.1t}$

when $t = 0$

$N = 500$

and $\lim\limits_{t \to \infty} N = 400$

Ans. A

Hence, initial population − limiting population

$= 500 - 400$

$= 100$

(2)

(4) Acute angle between $\underset{\sim}{i}+2\underset{\sim}{j}$ and $4\underset{\sim}{i}+2\underset{\sim}{j}$ Ans. D

let $\underset{\sim}{u} = \underset{\sim}{i}+2\underset{\sim}{j}$; $v = 4\underset{\sim}{i}+2\underset{\sim}{j}$

$$\cos\theta = \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|}$$

$$\cos\theta = \frac{4+4}{\sqrt{5}\times\sqrt{20}}$$

$$\cos\theta = \frac{8}{5\times2}$$

$$\cos\theta = \frac{4}{5}$$

$$\theta \simeq 36^\circ 52'$$

$$\theta \simeq 37^\circ$$

(5) $^{12}C_8 \times {}^4C_4$ Ans. C

(6) $\begin{pmatrix}4\\a+1\end{pmatrix}\cdot\begin{pmatrix}a\\-2\end{pmatrix} = 0$ Ans. B

$$4a - 2(a+1) = 0$$

$$2a - 2 = 0$$

$$a = 1$$

(7)

Ans. A

(8) Velocity vector is 45° below horizontal

Ans. B

when $\frac{\dot{y}}{\dot{x}} = -1$

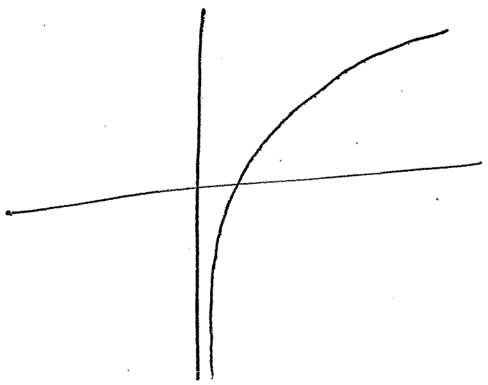
$\therefore \frac{-gt}{V} = -1$

$\therefore t = \frac{V}{g}$ seconds

(9) $g''(x) < 0$

Ans. C

Graph of $g(x)$



Ans. B

(10) $\sin(3x + x) - \sin(3x - x)$

$2\cos 3x \sin x$

Section II

Question 11

(a) $\dfrac{2x-1}{x+2} > 1$

$\dfrac{(2x-1)(x+2)^2}{(x+2)} > (x+2)^2$ and $x \neq -2$

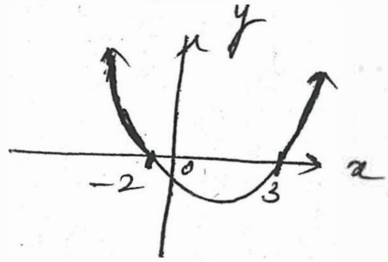
$(2x-1)(x+2) > (x+2)^2$

$(2x-1)(x+2) - (x+2)^2 > 0$

$(x+2)\{(2x-1)-(x+2)\} > 0$

$(x+2)(x-3) > 0$

$\therefore\ x < -2$ or $x > 3$



(b) let $u = 6 - x$ Evaluate $\int_1^6 x\sqrt{6-x}\, dx$

$$\int_1^4 x\sqrt{6-x}\, dx$$

$u = 6 - x$

$du = -dx$

limits:

$x = 1 \Rightarrow u = 5$

$x = 6 \Rightarrow u = 0$

$$= -\int_5^0 (6-u)\sqrt{u}\, du$$

$$= \int_0^5 (6-u)\sqrt{u}\, du$$

$$= \int_0^5 (6u^{1/2} - u^{3/2})\, du$$

$$= \left[\frac{6 \times 2}{3} u^{3/2} - \frac{2}{5} u^{5/2}\right]_0^5$$

$$= \left[4u^{3/2} - \frac{2}{5} u^{5/2}\right]_0^5$$

$$= 4 \times 5\sqrt{5} - \frac{2}{5} \times 5^2 \times \sqrt{5}$$

$$= 10\sqrt{5}$$

Question 11 - continued

(c) i, put 4 friends in class A, then need to choose 16 more students from 36

Probability all 4 friends in class A $= \dfrac{^{36}C_{16}}{^{40}C_{20}}$

could also be in class B

Probability all in same class $= 2 \times \dfrac{^{36}C_{16}}{^{40}C_{20}}$

3 d.p)

(ii) put 3 friends in class A

These 3 friends can be chosen in $^4C_3$ ways

then we need to choose 17 more students from 36 (because 4th friend can't be in class A)

Probability exactly 3 friends in class A $= \dfrac{^4C_3 \times {^{36}C_{17}}}{^{40}C_{20}}$

could also be in class B

Probability exactly 3 friends in same class $= 2 \times \dfrac{^4C_3 \times {^{36}C_{17}}}{^{40}C_{20}}$

$\simeq 0.499$ (3 d.p)

Question 12

(a)

$$\int_{\sqrt{2}}^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$$

$$= \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{2}}^{\sqrt{3}}$$

$$= \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\frac{1}{2}\right)$$

$$= \frac{\pi}{3} - \frac{\pi}{4}$$

$$= \frac{\pi}{12}$$

Question 12 - continued

(8)

(b) $x = 1 + 2\cos 2t$; $y = 2 + 2\sin 2t$

$x - 1 = 2\cos 2t \rightarrow (1)$ $\qquad$ $y - 2 = 2\sin 2t \rightarrow (2)$

squaring and adding eq (1) & (2)

$(x-1)^2 + (y-2)^2 = 4\cos^2 2t + 4\sin^2 2t$.

$(x-1)^2 + (y-2)^2 = 4(\sin^2 2t + \cos^2 2t)$

$\therefore$ $(x-1)^2 + (y-2)^2 = 4$ is the Cartesian equation

Question 12 continued

(c) let $p(n)$ be the proposition that $7^n - 3^n$ is divisible by 4 for $n \geq 1$

→ when $n = 1$; $LHS = 7^1 - 3^1 = 4$
$= 4 \times 1$

∴ $p(n)$ is true when $n = 1$

→ Assume that $p(n)$ is true when $n = k$
i.e, assume that $7^k - 3^k = 4M$ where $M \in \mathbb{Z}^+$
↳ eq (1)

→ Required to prove that $p(n)$ is true when $n = k+1$
i.e prove that $7^{k+1} - 3^{k+1} = 4P$, where $P \in \mathbb{Z}^+$

$L.H.S = 7^{k+1} - 3^{k+1}$

$= 7.7^k - 3.3^k$

$= 7(4M + 3^k) - 3.3^k$ (from eq (1), by assumption)

$= 7 \times 4M + 7.3^k - 3.3^k$

$= 7 \times 4M + 3^k(7-3)$

$= 7 \times 4M + 4.3^k$

$= 4(7M + 3^k)$

$= 4P$, where $P \in \mathbb{Z}^+$

∴ By principle of Mathematical induction if $p(k)$ is true
th $p(k+1)$ is true

Question 12 – continued

(d) Prove $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

$L.H.S = \cos 3\theta$

$= \cos(2\theta + \theta)$

$= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta$

$= (2\cos^2\theta - 1)\cos\theta - (2\sin\theta\cos\theta)\sin\theta$

$= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta(1 - \cos^2\theta)$

$= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta$

$= 4\cos^3\theta - 3\cos\theta$

$= R.H.S$, as required

Question 13

(a) If $t = \tan\frac{\theta}{2}$

(i) show that $\dfrac{1+\cos x+\sin x}{1-\cos x+\sin x} = \cot\dfrac{x}{2}$

$$1+\cos x+\sin x = 1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}$$

$$= \frac{(1+t^2)+(1-t^2)+2t}{1+t^2}$$

$$= \frac{2(1+t)}{1+t^2}$$

$$1-\cos x+\sin x = 1-\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}$$

$$= \frac{(1+t^2)-(1-t^2)+2t}{1+t^2}$$

$$= \frac{2t(1+t)}{1+t^2}$$

Hence $\dfrac{1+\cos x+\sin x}{1-\cos x+\sin x} = \dfrac{1}{t} = \cot\dfrac{x}{2}$.

Question 13 continued

(a) (ii) 12

Solve $\sin\theta + \cos\theta = \frac{1}{2}$ for $[0, 2\pi]$

$$\sin\theta + \cos\theta = \frac{1}{2}$$

$$\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = \frac{1}{2}$$

$$\frac{1+2t-t^2}{1+t^2} = \frac{1}{2}$$

$$2(1+2t-t^2) = 1+t^2$$

$$2+4t-2t^2 = 1+t^2$$

$$3t^2-4t-1=0$$

$$t = \frac{4 \pm \sqrt{16+4(3)(1)}}{6}$$

$$t = \frac{4 \pm \sqrt{28}}{6} = \frac{2 \pm 2\sqrt{7}}{3}$$

$$\tan\frac{\theta}{2} = \frac{2 \pm \sqrt{7}}{3}, \quad 0 \le \theta \le 2\pi; \quad 0 \le \frac{\theta}{2} \le \pi$$

$$\frac{\theta}{2} = \tan^{-1}\left(\frac{2 \pm \sqrt{7}}{3}\right) = 0.997, \ \pi - 0.212$$

$$= 0.997, \ 2.930 \text{ radians}$$

Test ~~of~~ for $\theta = \pi$ which is not a solution.

$\therefore \theta = 1.995, \ 5.859$ radians b, i, $\cos x - \sqrt{3}\sin x = R\cos(x+\alpha)$

$\cos x - \sqrt{3}\sin x = R\cos x\cos\alpha - R\sin x\sin\alpha$

Now comparing L.H.S & R.H.S
we can see that

$R\cos\alpha = 1$ ↳ ① and $R\sin\alpha = \sqrt{3}$ ↳ ②

squaring and adding eq ① & ②

$R^2\cos^2\alpha + R^2\sin^2\alpha = 1+3$

$R^2(\sin^2\alpha + \cos^2\alpha) = 4$

$R^2 = 4 \Rightarrow R = 2$ as $R > 0$

Since $\cos\alpha = \frac{1}{2} > 0$ and $\sin\alpha = \frac{\sqrt{3}}{2} > 0$

α is in the 1^{st} Quadrant and $\alpha = \frac{\pi}{3}$

$\therefore \cos x - \sqrt{3}\sin x = R\cos\left(x + \frac{\pi}{3}\right)$

Question 13 - Continued

14

b, ii, $\cos x - \sqrt{3}\sin x = 1$ for $[-\pi, \pi]$

$\therefore 2\cos\left(x + \frac{\pi}{3}\right) = 1$

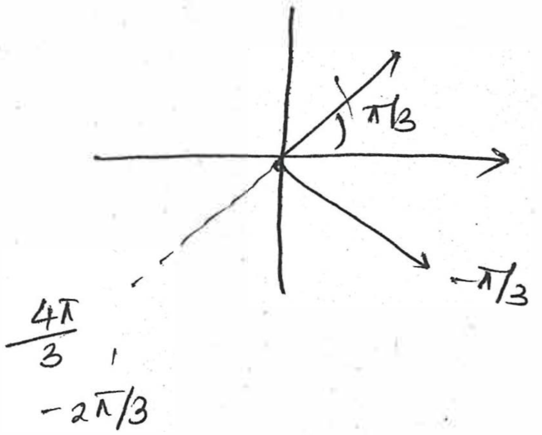
$\therefore \cos\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$

Changing the domain

$-\pi \le x \le \pi$

$\left(-\pi + \frac{\pi}{3}\right) \le \left(x + \frac{\pi}{3}\right) \le \left(\pi + \frac{\pi}{3}\right)$

$-\frac{2\pi}{3} \le \left(x + \frac{\pi}{3}\right) \le \frac{4\pi}{3}$



$\cos\left(x + \frac{\pi}{3}\right) = \frac{1}{2}$; related acute angle $= \frac{\pi}{3}$

$\therefore x + \frac{\pi}{3} = -\frac{\pi}{3}, \frac{\pi}{3}$

$\therefore x = -\frac{2\pi}{3}, 0$

Question 14

(a) α, β and γ are roots of $x^3-5x^2+7x+5=0$

i, $\alpha+\beta+\gamma = -\frac{b}{a}$

$= -(-5/1)$

$= 5$

ii, $\alpha\beta+\beta\gamma+\gamma\alpha = \frac{c}{a}$

$= \frac{7}{1}$

$= 7$

(b) $\underset{\sim}{u} = 2\underset{\sim}{i} + 3\underset{\sim}{j}$ and $\underset{\sim}{v} = -2\underset{\sim}{i} + 4\underset{\sim}{j}$

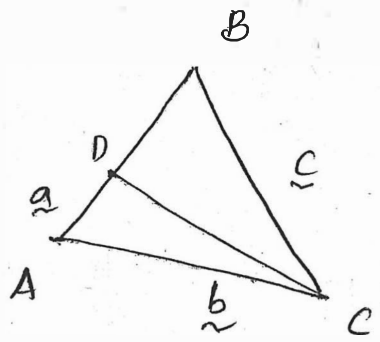
$\text{Proj}_{\underset{\sim}{u}} \underset{\sim}{v} = \frac{\underset{\sim}{u} \cdot \underset{\sim}{v}}{|\underset{\sim}{u}|^2} \cdot \underset{\sim}{u}$

$= \frac{2(-2)+(3\times4)}{2^2+3^2} \cdot \begin{pmatrix} 2 \\ 3 \end{pmatrix}$

$= \frac{8}{13}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ or $\frac{8}{13}(2\underset{\sim}{i}+3\underset{\sim}{j})$

Question 14 - continued

(c) Given $|\overrightarrow{AD}| : |\overrightarrow{DB}| = 2:3$



$\overrightarrow{AD} = \underset{\sim}{a}$

$\overrightarrow{AC} = \underset{\sim}{b}$

$\overrightarrow{CB} = \underset{\sim}{c}$

Show that $\underset{\sim}{b} = \frac{1}{2}(5\underset{\sim}{a} - 2\underset{\sim}{c})$

$|\overrightarrow{DB}| : |\overrightarrow{AD}| = 3:2$ and $\overrightarrow{AD} = \underset{\sim}{a}$

$$\overrightarrow{DB} = \frac{3}{2}\underset{\sim}{a}$$

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

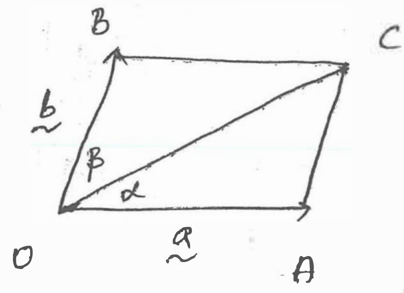
$$\underset{\sim}{b} = \overrightarrow{AD} + \overrightarrow{DB} - \overrightarrow{CB}$$

$$= \underset{\sim}{a} + \frac{3}{2}\underset{\sim}{a} - \underset{\sim}{c}$$

$$= \frac{5}{2}\underset{\sim}{a} - \underset{\sim}{c}$$

$$= \frac{1}{2}(5\underset{\sim}{a} - 2\underset{\sim}{c})$$

(d) i, Prove that $\underset{\sim}{a} \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = \underset{\sim}{b} \cdot (\underset{\sim}{a} + \underset{\sim}{b})$



$L \cdot H \cdot S = \underset{\sim}{a} \cdot (\underset{\sim}{a} + \underset{\sim}{b})$

$= \underset{\sim}{a} \cdot \underset{\sim}{a} + \underset{\sim}{a} \cdot \underset{\sim}{b}$

$= |\underset{\sim}{a}|^2 + \underset{\sim}{a} \cdot \underset{\sim}{b}$

$= |\underset{\sim}{b}|^2 + \underset{\sim}{a} \cdot \underset{\sim}{b}$ (Since OACB is a rhombus)

$= \underset{\sim}{b} \cdot \underset{\sim}{b} + \underset{\sim}{a} \cdot \underset{\sim}{b}$

$= \underset{\sim}{b} \, (\underset{\sim}{b} + \underset{\sim}{a})$

$= \underset{\sim}{b} \, (\underset{\sim}{a} + \underset{\sim}{b})$

$= R \cdot H \cdot S$

(d) ii Prove diagonal OC bisects $\angle AOB$.

from i, $\underset{\sim}{a} \cdot (\underset{\sim}{a} + \underset{\sim}{b}) = \underset{\sim}{b} \cdot (\underset{\sim}{a} + \underset{\sim}{b})$

$$|\underset{\sim}{a}| \cdot |\underset{\sim}{a} + \underset{\sim}{b}| \cos\alpha = |\underset{\sim}{b}| \times |\underset{\sim}{a} + \underset{\sim}{b}| \cos\beta$$

$$|\underset{\sim}{a}|\ |\underset{\sim}{a} + \underset{\sim}{b}| \cos\alpha = |\underset{\sim}{a}| \times |\underset{\sim}{a} + \underset{\sim}{b}| \cos\beta$$

$$\cos\alpha = \cos\beta \qquad (\text{since } |\underset{\sim}{a}| = |\underset{\sim}{b}|)$$

$$\alpha = \beta$$

$\therefore$ OC bisects $\angle AOB$

Question 15

(a) i, $\ddot{y} = -g$ $\ddot{x} = 0$

$\dot{y} = \int -g \, dt$ $\dot{x} = C_2$

Velocity: $\dot{y} = -gt + C_1$

when $t = 0$, $\dot{y} = \frac{15\sqrt{2}}{2}$

$\therefore \frac{15\sqrt{2}}{2} = -g(0) + C_1$

$\therefore C_1 = \frac{15\sqrt{2}}{2}$

and $\dot{y} = -gt + \frac{15\sqrt{2}}{2}$

when $t = 0$ $\dot{x} = \frac{15\sqrt{2}}{2}$

$\therefore C_2 = \frac{15\sqrt{2}}{2}$ and

$\dot{x} = \frac{15\sqrt{2}}{2}$

Displacement:

$y = \int \left(-gt + \frac{15\sqrt{2}}{2}\right) dt$

$y = \frac{-gt^2}{2} + \frac{15\sqrt{2}}{2} t + C_3$

when $t = 0$; $y = 30$

$\therefore 30 = -0 + 0 + C_3$

$\therefore C_3 = 30$ and

$y = \frac{-gt^2}{2} + \frac{15\sqrt{2}}{2} t + 30$

$x = \int \frac{15\sqrt{2}}{2} dt$

$x = \frac{15\sqrt{2}}{2} t + C_4$

when $t = 0$, $x = 0$

$\therefore C_4 = 0$ and

$x = \frac{15\sqrt{2}}{2} t$

using $g = 10$

$\therefore \underset{\sim}{s} = \left(\frac{15\sqrt{2}}{2} t\right) \underset{\sim}{i} + \left(-5t^2 + \frac{15\sqrt{2}}{2} + 30\right) \underset{\sim}{j}$

Question 15 - continued

aii, For the ball to reach the ground the component of $\underset{\sim}{j}$ must equal zero.

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$$-5t^2 + \frac{15\sqrt{2}}{2}t + 30 = 0$$

$$t = \frac{\frac{-15\sqrt{2}}{2} \pm \sqrt{\left(\frac{15\sqrt{2}}{2}\right)^2 - 4(-5)(30)}}{2(-5)}$$

$$= \frac{\frac{-15\sqrt{2}}{2} \pm \sqrt{712.5}}{-10}$$

$\simeq$ 3.7 seconds.

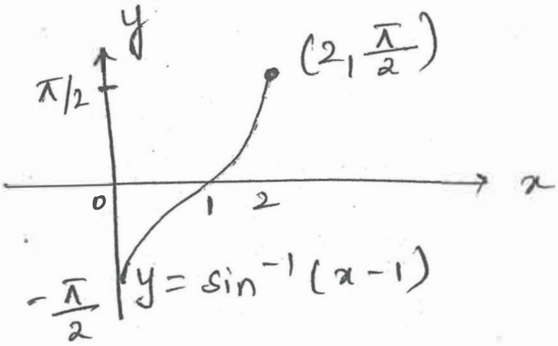
Question 15 – continued

(b) i, $f(x) = \sin^{-1}(x-1)$

Domain : $-1 \le (x-1) \le 1$

$0 \le x \le 2$

ii,



iii,

$$V = \pi \int_0^{\pi/2} (1+\sin y)^2 \, dy$$

$$= \pi \int_0^{\pi/2} (1 + 2\sin y + \sin^2 y) \, dy$$

$$= \pi \int_0^{\pi/2} (1 + 2\sin y + \tfrac{1}{2}(1-\cos 2y)) \, dy$$

$$= \pi \left[y - 2\cos y + \tfrac{1}{2}y - \tfrac{1}{4}\sin 2y \right]_0^{\pi/2}$$

$$= \pi \left[\tfrac{3}{2}y - 2\cos y - \tfrac{1}{4}\sin 2y \right]_0^{\pi/2}$$

$$= \pi \left[\left(\tfrac{3\pi}{4} - 0 - 0\right) - (0 - 2 - 0) \right]$$

$$= \pi \left(\tfrac{3\pi}{4} + 2 \right) \text{ cubic units}$$

Question 16

(a) constant term in $\left(x+\frac{2}{x}\right)^6$

$$T_{K+1} = {}^6C_K \, x^{6-K} \left(\frac{2}{x}\right)^K$$

$$= {}^6C_K \times 2^K \, x^{6-K} \left(\frac{1}{x^K}\right)$$

$$= {}^6C_K \times 2^K \, x^{6-2K}$$

for the constant term: $6-2K=0$

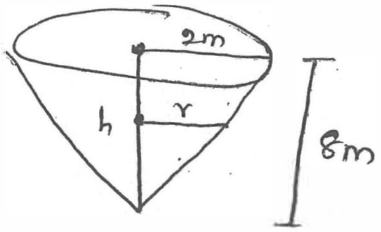
$\therefore K=3$

Hence, the constant term is ${}^6C_3 \times 2^3 = 160$

(b) i, Using similar triangles

$$\frac{2}{r} = \frac{8}{h}$$

$$\therefore r = \frac{h}{4}$$



Now $V = \frac{1}{3}\pi r^2 h$

$$V = \frac{1}{3}\pi \left(\frac{h}{4}\right)^2 \cdot h$$

$$V = \frac{\pi}{48} h^3$$

(b) ii) $V = \frac{\pi}{48} h^3$

(23)

$$\frac{dv}{dt} = \frac{dv}{dh} \times \frac{dh}{dt}$$

$$0.1 = \frac{d}{dh}\left(\frac{\pi}{48} h^3\right) \times 0.02$$

$$\frac{0.1}{0.02} = 3 \times \frac{\pi}{48} h^2$$

$$5 = \frac{\pi}{16} h^2$$

$$h^2 = \frac{80}{\pi}$$

Ce

$\therefore h \simeq 5.05\,m$

c i, Substituting $T = -40 + 64e^{kt}$ in $\frac{dT}{dt} = k(T - T_0)$

L.H.S : $\frac{dT}{dt}$

$= \frac{d}{dt}(-40 + 64e^{kt})$

$= 0 + 64ke^{kt}$

$= k(64e^{kt})$

$= k(T + 40)$

$= k(T - (-40))$

$= k(T - T_0)$, $T_0 = -40$

$=$ RHS

C II
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when $t=5$, $T=19$

$$19 = -40 + 64e^{5k}$$

$$e^{5k} = \frac{59}{64}$$

$$5K = \ln\left(\frac{59}{64}\right)$$

$$K = \frac{1}{5}\ln\left(\frac{59}{64}\right)$$

Sub $T=0$

$$0 = -40 + 64e^{kt}$$

$$40 = 64e^{kt}$$

$$e^{kt} = \frac{40}{64}$$

$$kt = \ln\left(\frac{40}{64}\right)$$

$$t = \frac{1}{k}\left(\ln\left(\frac{40}{64}\right)\right)$$

$$t = \frac{\ln(40/64)}{(1/5)\ln(59/64)}$$

$t \simeq 29$ seconds.